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LATE-TIME EVOLUTION OF DIRAC FIELD AROUND SCHWARZSCHILD-QUINTESSENCE BLACK HOLE

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The late-time evolution of Dirac field around spherically symmetric black hole surrounded by quintessence is studied numerically. Our results show, for lower values of the quintessence state parameter ϵ , Dirac field decays as power-law tail but with a slower decay rate than the corresponding Schwarzschild case. But for $\epsilon < -1/3$, all the ℓ -poles of the Dirac field give up the power-law decay form and relax to a constant residual field at asymptotically late times. The value of this residual field for which the field settles down varies on different surfaces. It has the lowest value on the black hole event horizon, increases as the radial distance increases and maximizes on the cosmological horizon.

Keywords: Black holes; Quintessence; Late-time tails

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1. Introduction

According to the black hole “*no-hair theorem*” a black hole formed by the gravitational collapse of a charged rotating star, will rapidly relaxes to the stationary state, characterized by three quantities, its mass, charge and angular momentum¹. Any other *hair* will disappear after the collapsing body settles down to its stationary state. Configurations violating the generalized no-hair conjecture were also presented including black holes dressed with Yang-Mills², Skyrme³ and dilaton⁴ fields, in various combinations with Higgs fields⁵. Even though most of these black holes are found to be unstable, there are few stable solutions also. By analyzing the stability of a black hole solution of the Einstein-Yang-Mills equations in the framework of small time-dependent perturbations it was shown that there is at least one exponentially growing radial mode with the correct boundary conditions at the horizon and at infinity⁶. It was proven that there are unstable modes of the

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Bartnik-McKinnon soliton and the non-abelian black hole solution of the Einstein-Yang-Mills theory for the gauge group $SU(2)$ ^{7,8}. The manner and rate with which the hairs of the black hole decay is thus an important question.

It was Richard Price⁹ who, making a perturbative analysis of the collapse of a nearly spherical star, showed that for a field with spin, s , any radiative multipole ($\ell \geq s$) gets radiated away completely, in the late stage of collapse. Further, he showed that at late times the field dies out with a power-law tail $t^{-(2\ell+p+1)}$, where $p = 1$, if the multipoles were initially static and $p = 2$ otherwise. Price's theorem was verified for various field perturbations around different black hole spacetimes in asymptotically flat spacetimes^{10,11,12,13,14,15,16,17,18}.

In the past two decades there have been growing observational evidences^{19,20} which established clearly that our universe is expanding in an accelerated pace, indicating the presence of some mysterious form of repulsive energy called dark energy. In order to explain the nature of dark energy, several models have been proposed. The simplest option being Einstein's cosmological constant(Λ), which has a constant equation of state with state parameter, $\epsilon = -1$, but it needs extreme fine tuning to account for the observations²¹. Models were proposed, replacing Λ with a dynamical, time-dependent and spatially inhomogeneous component now called quintessence, which can have an equation of state, $-1 \leq \epsilon \leq -1/3$ ^{22,23,24}. Quintessence hypothesis is found to fit current observations and more precise measurements may separate the two models in future. So it is interesting to check Price's theorem for a black hole with nonflat asymptotes.

For black holes in asymptotically de Sitter spacetimes, a generalization of Price's theorem was presented in²⁵, for massless small fluctuations. Later the classical black hole no-hair theorems were extended to spacetimes endowed with a positive Λ for different fields²⁶. In the perturbative study of Price's theorem, the late-time decay is determined by the asymptotic curvature of the spacetime²⁷. The studies on the time evolution of scalar, electromagnetic and gravitational perturbations, propagating on black hole with asymptotically de Sitter like spacetime revealed the existence of an exponentially decaying tails at late times contrasting the power-law tails in asymptotically flat situation^{29,30,31,32}.

The aim of this paper is to study the late-time evolution of Dirac field around a black hole whose asymptotes are determined by quintessence. For black hole with asymptotically flat spacetimes, the late-time behavior of Dirac field is well understood^{33,34,35}. The intermediate and late-time behavior of massive Dirac field, in the static spherically symmetric general black hole spacetime, is studied in³⁶. The late-time behavior of a massive Dirac field in the background of dilaton and brane-world black hole solutions is investigated in^{37,38}. It is revealed that for black hole in flat spacetimes, the long-lived oscillatory tail of massive Dirac field, decays as $t^{-5/6}$.

For Schwarzschild-de Sitter(SdS) black holes, the quasinormal modes(QNMs) of decay had been calculated for fields of different spin, including Dirac field³⁹. The QNMs of Dirac field around black holes surrounded by quintessence were calculated

in^{40,41}. But the late-time behavior of Dirac fields in these spacetimes is not clear. Recently, a proof of the no-hair theorem corresponding to perturbative massive spin-1/2 fields for stationary axisymmetric de Sitter black hole is presented⁴². So it is interesting to see how does Dirac field evolve in a spacetime in which the asymptotic structure is altered by the quintessence field.

The rest of the paper is organized as follows. In Sect.2 we introduce the master wave equation for Dirac field perturbations around black hole surrounded by quintessence. The numerical method used to study the time evolution is explained in Sect.3 and the results are presented. The conclusion and discussions are given in Sect.4.

2. Dirac field around black hole surrounded by quintessence

The exact solution of Einstein's equation for a static spherically symmetric black hole surrounded by the quintessential matter, under the condition of additivity and linearity in energy momentum tensor, was found in⁴³,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $f(r) = (1 - \frac{2M}{r} - \frac{c}{r^{3\epsilon+1}})$, M is the mass of the black hole, ϵ is the quintessential state parameter and c is the normalization factor which depends on the density of quintessence as, $\rho_q = \frac{-c}{2} \frac{3\epsilon}{r^{3(1-\epsilon)}}$. Various properties of this black hole solution were studied in the past (See Ref. 44, 45 for instance). It is difficult to analyze the perturbation in the above metric for arbitrary values of the parameter ϵ . For our study we take three special cases of the quintessence parameter $\epsilon = -1/3, -2/3$ and -1 , so that one can get simple expression for the radial tortoise coordinate in terms of the horizons and surface gravity associated with the horizons³² and the calculations become viable.

For $\epsilon = -1/3$, the black hole event horizon is located at $r_{e1} = 2M/(1-c)$. When $\epsilon = -2/3$, in addition to the black hole event horizon at $r = r_{e2}$ the spacetime possesses a cosmological horizon at $r = r_{c2}$, with $r_{e2} < r_{c2}$. The surface gravity associated with the horizons at $r = r_i$, is defined by $\kappa_i = \frac{1}{2}|df/dr|_{r=r_i}$ and we get,

$$\kappa_{e2} = \frac{c(r_{c2} - r_{e2})}{2r_{e2}}, \quad \kappa_{c2} = \frac{c(r_{e2} - r_{c2})}{2r_{c2}}. \quad (2)$$

The extreme case of quintessence, $\epsilon = -1$, corresponds to the SdS spacetime. The surface gravity at the event horizon, $r = r_{e3}$ and the cosmological horizon $r = r_{c3}$, are given by,

$$\kappa_{e3} = \frac{c(r_{c3} - r_{e3})(r_{e3} - r_0)}{2r_{e3}}, \quad \kappa_{c3} = \frac{c(r_{c3} - r_{e3})(r_{c3} - r_0)}{2r_{c3}}, \quad (3)$$

where the third root of the polynomial equation $f(r) = 0$, $r_0 = -(r_{e3} + r_{c3})$, with $r_0 < r_{e3} < r_{c3}$.

The Dirac equation for a massless field in spacetime $g_{\mu\nu}$, specified by Eq.(1) has the form⁴⁶,

$$i\gamma^a e_a^\mu (\partial_\mu + \Gamma_\mu) \Psi = 0, \quad (4)$$

where γ^a are the Dirac matrices, Γ_μ are the spin connections and e_a^μ are the tetrad.

The radial part of the above perturbation equations can be decoupled from their angular parts and reduced to the form^{47,40},

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} \right) \Psi_\ell(t, r) = -V_\pm(r) \Psi_\ell(t, r), \quad (5)$$

where the tortoise coordinate is defined by, $dr_* = (1/f)dr$ and the effective potentials are given by,

$$V_\pm = \frac{|k|\sqrt{f}}{r^2} \left[|k|\sqrt{f} \pm \frac{r}{2} \frac{\partial f}{\partial r} \mp f \right], \quad (6)$$

where k is a positive or a negative nonzero integer related to the total orbital angular momentum by $k = \ell + 1$ for (+) sign and $k = \ell$ for (-) sign. The potentials, V_+ and V_- are the super symmetric partners and give same spectra⁴⁸, so we choose V_+ by omitting the subscripts.

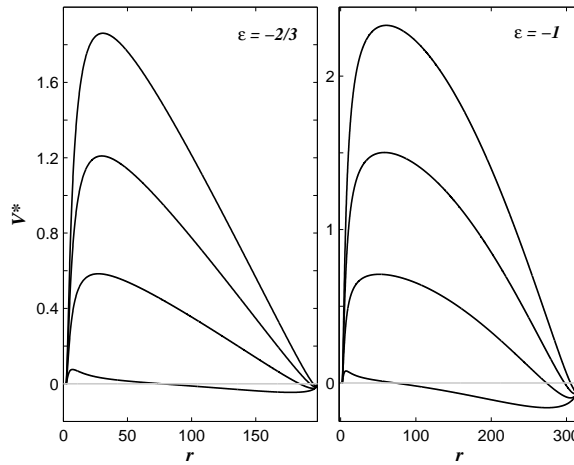


Fig. 1. Plots of effective potentials experienced by the Dirac field for quintessence parameters $\epsilon = -2/3$ with $c = 10^{-2}/2$ and $\epsilon = -1$ with $c = 10^{-5}$. Curves from bottom to top are for $\ell = 0, 1, 2$ and 3 modes. The potential is scaled as $V^* = V(r_e - r)^2(\ell + 1/2)$, to enhance the features at large r . Potentials have a negative dip near cosmological horizon.

For $\epsilon = -1/3$, the effective potential is positive definite for $r_* \in [-\infty, +\infty]$ and has a potential barrier near the event horizon but vanishes asymptotically as $r_* \rightarrow \pm\infty$. As the parameter ϵ decreases below $-1/3$, a cosmological horizon is created by the quintessence. Figure 1 shows effective potentials for $\epsilon = -1$ and $-2/3$. In these cases, after a barrier nature near the event horizon, the effective potentials for all modes, vanish at some $r_* = r_*^0$ and there after form a negative well in the range $r_*^0 < r_* < +\infty$. This behavior of Dirac field is in contrast with other fields. For scalar field, even if the $\ell = 0$ mode shows the negative dip in the potential, all other higher modes have a positive value for the potential between the horizons^{32,29}.

3. Numerical integration and results

The complex nature of the potentials makes it difficult to obtain the exact solutions of Eq.(5) and we have to tackle the problem by numerical methods. A simple and efficient method to study the evolution of field were developed in¹⁰, after recasting the wave equation, Eq.(5), in the null coordinates, $u = t - r_*$ and $v = t + r_*$ as,

$$-4 \frac{\partial^2}{\partial u \partial v} \Psi(u, v) = V(u, v) \Psi(u, v) \quad (7)$$

and using the following discretization,

$$\Psi_N = \Psi_W + \Psi_E - \Psi_S - \frac{h^2}{8} V(S) (\Psi_W + \Psi_E) + O(h^4). \quad (8)$$

The numerical integration is performed on an uniformly spaced grid with points, $N(u+h, v+h)$, $W(u+h, v)$, $E(u, v+h)$ and $S(u, v)$ forming a null rectangle with an overall grid scale factor of h . The tortoise coordinates are inverted using Newton-Raphson method³². The field is scaled as, $\phi = \psi/r$ and the evolution is monitored on different surfaces viz.,

- (1) the cosmological horizon (approximated by the null surface, $v = v_{max}$),
- (2) the black hole event horizon (approximated by the null surface, $u = u_{max}$) and
- (3) different null surfaces of fixed radius, $r_* = K$.

Figure 2 shows the evolution profile of the Dirac field around the black hole in a quintessence filled universe along with that in the pure Schwarzschild space-time. Monopole and dipole fields are shown. We observe that the evolution shows deviations from the Schwarzschild case, after initial transient phase. The damped oscillation(QNM) phase and the late-time tail of decay in the final phase (these two phases depend only on the characteristics of the background black hole spacetime⁴⁹) show the signature of the quintessence. We observe that the field decays slowly in the QNM phase, if quintessence is present as it was shown in^{39,40}, using the WKB method. The QNM phase is followed by the regime of late-time tails of field decay.

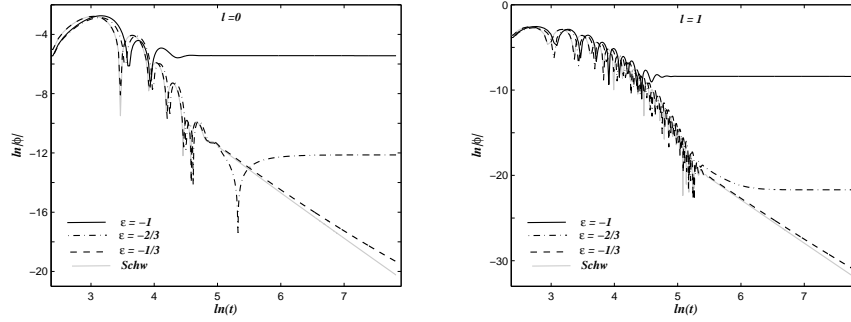


Fig. 2. Log-log graph of the evolution of Dirac field in a quintessence filled black hole spacetime with $c = 10^{-2}/2$, in comparison with that in the pure Schwarzschild spacetime, evaluated at $r^* = 10$. $\ell = 0$ and $\ell = 1$ modes for $\epsilon = -1/3, -2/3$ and -1 .

It is well known that, the field has the power-law decay in the pure Schwarzschild spacetime, represented by the straight lines in a log-log plot. Power-law decay is observed for the $\epsilon = -1/3$ case of quintessence, but with slightly slower decay rate than the corresponding Schwarzschild tail. For $c = 10^{-2}/2$, we get $\phi \sim t^{-(2\ell+2.7)}$, a slower decay than the $\phi \sim t^{-(2\ell+3)}$ of the pure Schwarzschild case.

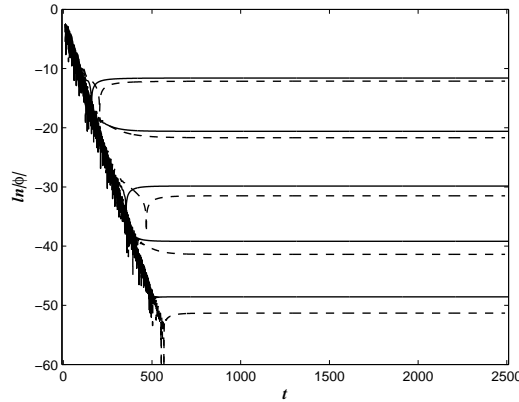


Fig. 3. Evolution of Dirac field extracted at $r^* = 10$ for the $\epsilon = -2/3$ case with $c = 10^{-2}/2$ (solid curves) and $\epsilon = -1$ case with $c = 10^{-5}$ (dashed curves). In each case, curves from top to bottom are for $\ell = 0, 1, 2, 3$ and 4 .

Figure 3 shows the evolution profile for the $\epsilon = -2/3$ and $-1/3$ cases. At late times, the field does not decay for these cases. All the ℓ -poles of field relaxes to a constant residual field, ϕ_0 at asymptotic late times. This behavior of Dirac field is in contrast with other spin fields, for which an exponential decay was observed^{29,31,32}. Similar behavior was observed for the monopole of the scalar field but all the $\ell > 0$

modes were found to be exponentially decaying. The nature of Dirac field is little surprising and it strengthens the dependence of the unusual negative dip in the potential near cosmological horizon and the relaxation of the field to a constant value. In order to verify that the residual static field is not a numerical error we checked the convergence of the code by decreasing the grid space and can find a good convergence as $h \rightarrow 0$.

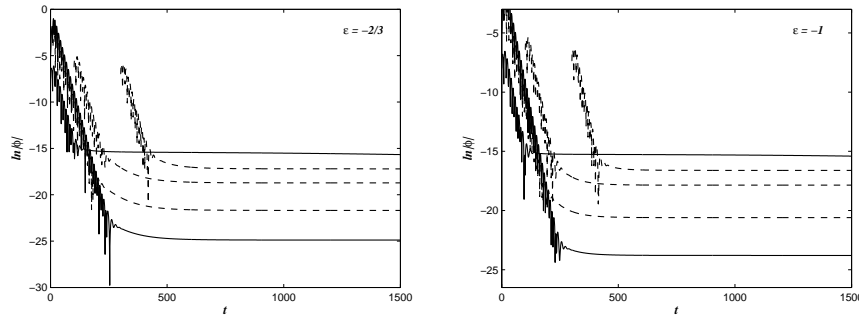


Fig. 4. The decay of $\ell = 2$ mode of Dirac field on different surfaces. Solid curves represent the field on event horizon(bottom) and cosmological horizon(top). Dotted curves from bottom to top correspond to the field on surfaces at $r^* = 10, 100$ and 300 .

To confirm that the behavior of Dirac field is not an artifact of the particular location, we monitor the evolution of the field on different null surfaces of constant radius and on the event and cosmological horizons. Figure 4 shows the decay of $\ell = 2$ mode of Dirac field on the black hole event horizon, cosmological horizon and three surfaces of fixed radius, $r^* = 10, 100$ and 300 . The constant asymptotic value of the Dirac field, ϕ_0 , varies from the black hole event horizon to the cosmological horizon. The ϕ_0 has a lowest value on the event horizon, increases as radial position goes farther, and has the highest value on the cosmological horizon.

The dependence of the asymptotic residual field, ϕ_0 , on the parameter c , is shown in Fig.5, in logarithmic scale. For $\epsilon = -1$, a least square fit for $\ln|\phi_0| = m \ln(c) + c1$, gives the slopes 0.973, 1.933, 2.942, 3.915 and 4.921 for $\ell = 0, 1, 2, 3$ and 4, respectively. The y intercepts are 1.921, 4.288, 7.470, 8.646 and 10.716. For $\epsilon = -2/3$, we get the slopes 1.768, 3.630, 5.559, 7.330 and 9.625 and y intercepts -0.445, -0.423, -0.519, -0.897 and -0.744, for $\ell = 0, 1, 2, 3$ and 4, respectively. These results suggest that,

$$\begin{aligned}
 \phi_0 &\sim c^{(\ell+1)}, & \text{for } \epsilon = -1, \\
 \phi_0 &\sim c^{1.782(\ell+1)}, & \text{for } \epsilon = -2/3.
 \end{aligned} \tag{9}$$

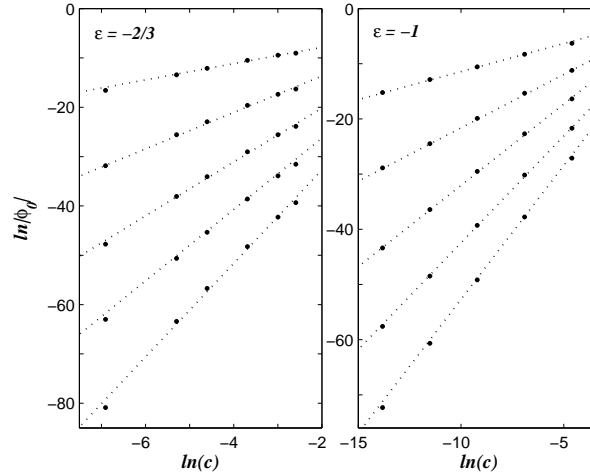


Fig. 5. The asymptotic value of the Dirac field along event horizon, ϕ_0 plotted against c , in logarithmic scale. Dotted lines represent a linear fit. Curves from top to bottom are for $\ell = 0, 1, 2, 3$ and 4.

4. Conclusions and discussions

The paper studies the evolution of Dirac field perturbation, particularly the late-time behavior, around a black hole spacetime surrounded by quintessence. The quintessence equation of state, ϵ , plays a dramatic role in the late-time decay of the Dirac field. For $\epsilon = -1/3$ the late-time decay follows a power-law form, but with a lower decay rate than the corresponding Schwarzschild case. As the value of the quintessential parameter ϵ , decreases the cosmological horizon forms and a negative dip appears in the effective potential near the cosmological horizon. As it seems to be the consequence of the peculiar behavior of potential, for $\epsilon = -2/3$ and -1 , the Dirac field does not decay to zero, but relaxes to a constant residual field, at late times. The values of the residual field is determined by the values of the parameter ϵ and c .

The asymptotic value of the Dirac field varies on different surfaces of constant radius. It has the lowest value on the black hole event horizon, increases as the radial distance increases and maximizes on the cosmological horizon. This behavior of Dirac field seems to be odd comparing with other spin field perturbations, where all the $\ell > 0$ modes of the field decay exponentially. Price's original work demonstrates that there can be no static solution to the scalar wave equation that are well behaved at infinity and at black hole event horizon. Even though the $\ell = 0$ mode of the scalar field in the SdS spacetime is observed to settle down to a constant asymptotic value, it relaxes to the *same* constant value on all the surfaces. It can be argued that the constant field does not carry any stress energy tensor and it is equivalent to vanishing of the hair. But the behavior of Dirac field is rather intriguing since all

the ℓ modes of the field have non zero value at late times and it varies on different surfaces. Our study indicates that there may be static solutions for Dirac field for all ℓ for black holes with de-Sitter like asymptotes. The presence non decaying wave tails at late times may lead to the instability of Cauchy horizons inside charged and rotating black holes and the strong cosmic censorship has to be revisited⁵⁰ for these spacetimes. Further detailed numerical and analytical studies of the spin-1/2 fields around black holes in an expanding universe are, hence, call forth.

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